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A Note on Riviere, S Conjecture

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Abstract

In this note,we consider the equations $\Delta u = \Omega \nabla u$ with the unknown vector $u \in W^{1,2}\left(B_1^n, \mathbf{R}^m\right)$. here $\Omega = \left(\Omega_j^i\right)_{1 \le i,j \le m} \in L^2\left(\mathbf{R}^n, \mathbf{R}^m\right)$ If there exists a matrix valued function $A, A^{-1} \in W^{1,2}\left(\mathbf{R}^n\right) ? L^\infty\left(\mathbf{R}^n\right)$,such that $\Omega = -A \nabla A^-$, then the solution $u \in W^{2,1}\left(B_{1/4}^n\right)$. In particular, the equations $\Delta u = \Omega \nabla u$ with $u, \theta \in W^{2,1}\left(B_1^n, \mathrm{R}^m\right)$, $\Omega = \begin{pmatrix} 0 & \nabla\theta \\ -\nabla\theta & 0 \end{pmatrix}$, then the solution $u \in W^{2,1}\left(B_{1/4}^n\right)$.

Keywords

Conservation laws; Non-linear elliptic systems; Harmonic maps 2010 MR Subject Classification 00A99; 97U60; 97U70.

1. Introduction

We consider the system:

$$\Delta u^i = \Omega_j^i \nabla u^j, i = 1, 2, \cdots, m \tag{1}$$

here $u \in W^{1,2}\left(\mathbf{R}^n\right)$,i=1,2,⋯,m .and $\Omega_j^i \in L^2\left(\mathbf{R}^n, \mathbf{R}^m\right), i, j = 1, 2, \cdots, m$. If we write $u = \left(u^1, u^2, \cdots, u^m\right), \Omega = \left(\Omega_j^i\right)_{1 \le i, j \le m}$, the system can be written

$$\Delta u = \Omega \nabla u \tag{2}$$

This is an important PDEs in geometric variational problems. For example, for harmonic maps from B_1^n into N^k, k—dimensional closed submanifold of R^m, the equation is

$$\Delta u = A(u)(\nabla u, \nabla u) \tag{3}$$

here A is the second fundamental form of N^k in R^m. Using the definition of A, we see that (using that $\nabla u^k \nu_i^k = 0$ for every i, here (ν_{n+1}, ...,ν_m) be a smooth local orthogonal frame for the normal bundle near u(x))

$$\begin{aligned} \Delta u^s &= \sum_{i,k} \left(\nabla u, \left(d_k \nu_i\right)(u) \nabla u^k\right) \nu_i^s(u) \\ &= \sum_{i,k,l} \nabla u^k \left(\nu_i^s(u)\left(d_k \nu_i\right)^l - \nu_i^k(u)\left(d_s \nu_i\right)^l\right) \end{aligned}$$

and hence u solves an equation of the form (2) with

$$\Omega_k^s = \sum_{i,l} \left(\nu_i^s(u)\left(d_k \nu_i\right)^l - \nu_i^k(u)\left(d_s \nu_i\right)^l\right)$$

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In this note, we try to proof $u \in W_{loc}^{2,1}$ .Notice that the right side of the equation (2) is in L^1, we can't get it from Caldron-Zygmund's estimate. But if div Ω= 0, we can be proved that $\Omega\nabla u$ is in H^1, a subspace of L^1, and can deduce that $u \in W_{loc}^{2,1}$. For systems, even div Ω≠ 0, it can be proved the same result under some conditions. T.Riviere [2] proved $u \in W_{loc}^{2,1}$ of the equations (2) for n = 2, with $\Omega_i^i = -\Omega_i^j$,hence $u \in C_{loc}^0$. Using this result, he can prove some geometric variational equations such as harmonic maps from surface to arbitrary manifold are smooth. The key step is to express the system (2) in divergence form. The procedure consists of finding a map A which takes values in the space of m ×m-invertible matrices and a two-vector field tensored with m×m-matrices B satisfying

$$\nabla A - A\Omega = curl\,\mathrm{B},$$

where curlB is the matrix valued vector fields. Once the existence of the solution regular enough, A and B satisfying (2) has been established, one observes that the system (2) is equivalent to the following conservation law

$$div(A\nabla u + B\cdot\nabla u) = 0.$$

For n > 2, the nature question is: Is the solutions of [2] still in $W_{loc}^{2,1}$? or is every weak harmonic maps $u \in W^{1,2}$ from B_2^n into N^k, k-dimensional closed submanifold of R^m in $W^{2,1}$ (B_1^n)? The later question is link to the quantization result of [3] to general targets.

Using the similar method, we will give a partial answer to the first question, it may also useful to study the second question, which is a conjecture on T.Riviere's paper [2].

Theorem 1.1 Suppose $u \in W^{1,2}\left(B_1^n, \mathbf{R}^m\right)$. If there exists a matrix valued function A, $A, A^T \in W^{1,2}\left(\mathbf{R}^n\right)$? $L^\infty\left(\mathbf{R}^n\right)$, such that

$$\Omega = -A\nabla A^{-1},$$

then the solution of (2) $u \in W^{2,1}\left(B_{1/4}^n\right)$.

Remark 1.1 In this case,Ω needn't be antisymmetric.

Corollary 1.2 Suppose $u \in W^{1,2}\left(B_1^n, \mathrm{R}^m\right)$. If there exists a matrix valued function A, $A, A^T \in W^{1,2}\left(\mathbf{R}^n\right)$? $L^\infty\left(\mathbf{R}^n\right)$ with AAT = I, such that

$$\Omega = -A\nabla A^T,$$

then the solution of (2) $u \in W^{2,1}\left(B_{1/4}^n\right)$.

Remark 1.3 In this case,Ω is antisymmetric, that is $\Omega_i^i = -\Omega_i^j$, i,j = 1, 2, … ,m.

Corollary 1.4 Suppose $u \in W^{1,2}\left(B_1^n, \mathrm{R}^2\right)$. If there exists a function $\theta \in W^{1,2}\left(\mathrm{R}^n\right)$ such that

$$\Omega = \begin{pmatrix} 0 & \nabla\theta \\ -\nabla\theta & 0 \end{pmatrix}$$

then the solution of (2) $u \in W^{2,1}\left(B_{1/4}^n\right)$.

Remark 1.5 In Corollary 1.2 if write f = u1 + iu2, the equations becomes

$$\Delta f = i\nabla\theta\nabla f.$$

2. Preparation

In order to prove this results, we need some lemmas.

2.1. Hardy Space and Div-curl Lemma

Let Φ be a Schwartz function with $\int_{\mathrm{R}^n}\Phi = 1$, for tempered distributions f, define

$$\left(M_\Phi f\right)(x) = \sup_{t>0}\left|\left(f * \Phi_t\right)(x)\right|,$$

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where * is convolution and $\Phi_t(x)=t^{-n}\Phi\left(\frac{x}{t}\right)$. Define $f\in\mathcal{H}^1\left(\mathbf{R}^n\right)$ if $\left(M_\Phi f\right)(x)\in L^1\left(\mathbf{R}^n\right)$. The $\mathcal{H}^1$ -norm $\|f\|_{\mathcal{H}^1}$ of a distribution f is defined to be the L1 norm of $\left(M_\Phi f\right)(x)$ (this depends on the choice of Φ, but different choices of Schwartz functions Φ give equivalent norms). $\mathcal{H}^1$ space is a strictly subspace of L1 space.

In the next lemma (which was proved by Coifman, Lions, Meyer and Semmes in [6]) we give a very important example of a function belonging to $\mathcal{H}^1$.

Lemma 2.1 Suppose $E\in L^p\left(\mathbf{R}^n,\mathbf{R}^n\right)$ $divE=0$; $B\in W^{1,q}\left(\mathbf{R}^n\right)$, $p,q>1,\frac{1}{p}+\frac{1}{q}=1$, then $E\cdot\nabla B\in\mathcal{H}^1\left(\mathbf{R}^n\right)$, and $\|E\cdot\nabla B\|_{\mathcal{H}^1}\le C(p)\|E\|_{L^p}\|\nabla B\|_{L^q}$.

2.2. Biot-savart Law in High Dimension

Lemma 2.2 (Biot-Savart law in high dimension)

Suppose div v = 0, Ω= curl $v\in\mathcal{H}^1\left(R^n\right)$,let

$$\$=\frac{1}{w_n}\int_{R^n}\frac{\Omega(y)(x-y)}{|x-y|^n}dy$$

then

$$\|\nabla\$\|_{\mathcal{H}^1}\le C\|\Omega\|_{\mathcal{H}^1},$$

and $v-\$$ is harmonic, here w_n denotes the area of the surface $\left\{x\in R^n:|x|=1\right\}$.

Proof Notice that

$$\begin{aligned}\nabla_i\$_j-\nabla_j\$_i&=\frac{1}{\omega_n}\sum_k\int_{\mathbf{R}^n}\frac{\left(\nabla_i\Omega(y)_{j,k}-\nabla_j\Omega(y)_{i,k}\right)\left(x_k-y_k\right)}{|x-y|^n}dy\\&=\frac{1}{\omega_n}\sum_k\int_{\mathbf{R}^n}\frac{\nabla_k\Omega(y)_{j,i}\left(x_k-y_k\right)}{|x-y|^n}dy\\&=\frac{1}{\omega_n}\lim_{\varepsilon\to0}\sum_k\int_{|x-y|>\varepsilon}\frac{\nabla_k\Omega(y)_{j,i}\left(x_k-y_k\right)}{|x-y|^n}dy\\&=\frac{1}{\omega_n}\lim_{\varepsilon\to0}\sum_k\int_{|x-y|>\varepsilon}\Omega(y)_{j,i}\nabla_k\frac{\left(x_k-y_k\right)}{|x-y|^n}dy\\&\quad+\frac{1}{\omega_n}\lim_{\varepsilon\to0}\sum_k\int_{|x-y|=\varepsilon}\Omega(y)_{j,i}\frac{\left(x_k-y_k\right)^2}{|x-y|^{n+1}}dS\\&=\Omega(y)_{j,i}\end{aligned}$$

and

$$div\$=\frac{1}{\omega_n}\sum_i\sum_k\int_{\mathbf{R}^n}\frac{\nabla_i\Omega(y)_{i,k}\left(x_k-y_k\right)}{|x-y|^n}dy=0$$

we have

$$div(v-\$)=0$$

$$curl(v-\$)=0$$

there exist a function ψ ,such that $v-\$=\nabla\psi$. and $\Delta\psi=0$ = 0, which means that $v-\$$ is harmonic. Moreover, it can be proved that (for example [5], p74-75)

$$\nabla v=P(\Omega)+C\Omega$$

here P is a singular integral operator and C is a constant matrix.

By singular integral theory (for example [1]) we know that

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$$\|\nabla \tilde{v}\|_{\mathcal{H}^1} \le C\|\Omega\|_{\mathcal{H}^1}.$$

3. Proof

Proof of the Theorem.
First, we need to write (2) in divergence form.

$$div\left(A^{-1}\nabla u\right) = \nabla A^{-1}\nabla u + A^{-1}\Delta u$$
$$= A^{-1}\left(A\nabla A^{-1}\nabla u + \Delta u\right) = 0.$$

by Hodge decomposition, suppose

$$A^{-1}\nabla u = \Delta\alpha + \beta \tag{4}$$

where divβ = 0. we have

$$\Delta\alpha = 0. \tag{5}$$

denote $A^{-1} = \left(A^{-1}_{s,k}\right)_{2\times 2}$, curlβs denotes the the curl of the vector ps,i t is an anti-symmetic matrix,then

$$\mathrm{curl}\,\beta^s_{i,j} = \mathrm{curl}\left(\sum_{k=1,2} A^{-1}_{s,k}\nabla u^k\right)_{i,j}$$
$$= \sum_{k=1,2}\nabla_i\left(A^{-1}_{s,k}\nabla_j u^k\right) - \sum_{k=1,2}\nabla_j\left(A^{-1}_{s,k}\nabla_i u^k\right)$$
$$= \sum_{k=1,2}\left(\nabla_i A^{-1}_{s,k}\nabla_j u^k - \nabla_j A^{-1}_{s,k}\nabla_i u^k\right)$$

Now let $\tilde{u} \in W^{1,2}_0\left(\mathbf{R}^n, \mathbf{R}^2\right)$ be the Sobolev extension of u, and $\nabla\tilde{u} = \nabla u$ on $B_{1/2}$. Moreover,

$$\|\nabla\tilde{u}\|_{L^2\left(\mathbf{R}^n\right)} \le C\|\nabla u\|_{L^2\left(B_{1/2}\right)}$$

We extend A-1 in the same way and denote the resulting map by $\tilde{A}^{-1}$. Denote

$$\mathrm{curl}\,\tilde{\beta}^s_{i,j} = \sum_{k=1,2}\left(\nabla_i\tilde{A}^{-1}_{s,k}\nabla_j\tilde{u}^k - \nabla_j\tilde{A}^{-1}_{s,k}\nabla_i\tilde{u}^k\right)$$

by lemma 2.1, we know

$$\mathrm{curl}\,\tilde{\beta} \in \mathcal{H}^1.$$

Let

$$\gamma = \frac{1}{\omega_n}\int_{\mathbf{R}^n}\frac{\mathrm{curl}\,\tilde{\beta}(y)(x-y)}{|x-y|^n}dy,$$

as in lemma 2’s proof, it can be proved that
divγ = 0
curlγ = curl $\tilde{\beta}$
By lemma 2.2, we have γ -β is harmonic and

$$\nabla\gamma \in L^{-1} \tag{6}$$

Now restricted on B1/4, together with (4), (5), (6) and γ -β is harmonic, we have

$$u \in W^{2,1}\left(B^n_{1/4}\right).$$

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Proof of Corollary 1.2.

Let

$$P(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

direct compute shows that

$$P(\theta)^{-1}\nabla P(\theta)+\begin{pmatrix} 0 & \nabla\theta \\ \nabla\theta & 0 \end{pmatrix}=0$$

Notice that $P(\theta)P(\theta)^T=I$ together with corollary 1.1(It is obviously) gives Corollary 1.2.

References

[1] C. Fefferman and E. Stein. Hp spaces of several variables. Acta Math., 129,137-193,1972.

[2] T. Riviere. Conservation laws for conformally invariant variational problems. Invent. Math., 168,1-22, 2007.

[3] F-H,Lin; T.Riviere. Energy quantization for harmonic maps. Duke Math. J. 111,no. 1, 177—193,2002.

[4] H. Wente. An existence theorem for surfaces of constant mean curvature. J. Math. Anal. Appl., 26,318-344, 1969.

[5] Majda;A. Bertozzi. Vorticity and incompressible flow. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002

[6] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. J. Math. Pures Appl., 72,247-286, 1993.